A non-local, Lorentz-invariant, hidden-variable interpretation of relativistic quantum mechanics based on particle trajectories

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Abstract
We demonstrate how to construct a Lorentz-invariant, hidden-variable interpretation of relativistic quantum mechanics based on particle trajectories. The covariant theory that we propose employs a multi-time formalism and a Lorentz-invariant rule for the coordination of the spacetime points on the individual particle trajectories. In this way we show that there is no contradiction between non-locality and Lorentz-invariance in quantum mechanics. The approach is illustrated for relativistic bosons, using a simple model to discuss the individual non-locally correlated particle motion which ensues when the wavefunction is entangled. A simple example of measurement is described.

PACS numbers: 03.65.Pm, 03.65.Vd, 03.65.Ta, 03.70.+k

1. Introduction

In non-relativistic de Broglie–Bohm theory, when a many-particle wavefunction is entangled the individual particle trajectories are non-locally correlated. That is, the velocity of a given particle depends not just on its own position but also on the simultaneous positions of all the other particles at time t as well

$$\vec{v}_i = V(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n, t). \tag{1}$$

Thinking of the particles as instantaneously, non-locally correlated one wonders what happens in relativistic theory where time-ordering is not unique for space-like separated events. In which frame does one consider the particles at the same time? There seems to be an apparent conflict between non-locally correlated many-particle trajectories and the demands of relativistic covariance. Hardy [1] has argued that any hidden-variable approach to quantum theory will not satisfy Lorentz-invariance at the level of the hidden variables, although the statistical predictions of relativistic quantum theory clearly will do so. Clearly the issue can only be resolved properly within a covariant relativistic hidden-variable theory for many

particles. On the whole such relativistic formulations as exist [2, 3] (but also see [4–6]) simply formulate the theory in a given frame.

As far as we know, although de Broglie formulated a theory for the Klein–Gordon equation [7], he did not explicitly discuss this issue, whereas Bohm suggested that his approach indicated the need for a preferred inertial frame, with Lorentz-invariance being satisfied *only* at the level of the statistical predictions of the theory. Valentini [8] has suggested that the search for a Lorentz-invariant hidden-variable theory is in any case misguided as the fundamental symmetry of the de Broglie–Bohm theory should be seen as Aristotlean. Because our purpose in this paper is to present an explicitly covariant de Broglie–Bohm theory, we implement the model for the case of massive bosons, but the fundamental principle can be used for fermion theories as well.

Several alternative ontologies have been proposed in the context of de Broglie–Bohm theories. Bohm believed that bosons and fermions have a fundamentally different character; in his approach, bosons are true fields whereas fermions are true particles. Both fields and particles are well-defined and evolve in a continuous and causal manner according to deterministic equations of motion. Valentini [9] has suggested a theory in which both fermions and bosons are to be described by well-defined fields. de Broglie preferred an ontology of particles for both bosons and fermions, but for the boson case his prescription of the particle trajectories had some serious pathologies. Elsewhere [10], we have discussed an outline of an approach which overcomes some of the difficulties inherent in de Broglie's theory. More recently [11] we developed the approach in more detail, demonstrating that it is possible to develop a de Broglie–Bohm particle trajectory approach for the Klein–Gordon equation which avoids the pathologies of de Broglie's original theory. de Broglie's guidance condition for scalar bosons was based on the charge current, whereas our approach uses the time-like flows of stress–energy–momentum to define particle trajectories which follow the flows.

2. Stress–energy–momentum tensor for many particles

Our Lorentz-invariant description defines the flow of stress–energy–momentum, and hence particle trajectories which follow the flow, through the intrinsic natural four-vector provided by the matter field itself through the eigenvalue equation

$$T_\nu^\mu W^\nu = \lambda W^\mu \tag{2}$$

where T_ν^μ is the stress–energy–momentum tensor, λ the eigenvalue and W^μ the eigenvector. If ϕ is a solution to the Klein–Gordon equation then

$$T_{\mu\nu} = \frac{1}{2} \left[\phi_{,\mu}^* \phi_{,\nu} - g_{\mu\nu} \left(\phi^{,\sigma} \phi_{,\sigma} - m^2 \phi^* \phi \right) + \phi_{,\mu} \phi_{,\nu}^* \right]. \tag{3}$$

Writing the solution ϕ as

$$\phi = \exp[P + iS] \tag{4}$$

the stress–energy–momentum tensor, T_ν^μ, of the field ϕ is given by

$$T_\nu^\mu = |\phi|^2 \left[m^2 - (P^\alpha P_\alpha + S^\alpha S_\alpha) \right] \delta_\nu^\mu + 2|\phi|^2 \left[(P^\mu P_\nu + S^\mu S_\nu) \right]. \tag{5}$$

For an n-particle system with no inter-particle interactions, the Lagrangian density can be taken to be

$$L = \sum_i \frac{1}{2} \left(\phi^{,\mu_i} \phi_{,\mu_i}^* - m_i^2 \phi \phi^* \right) \tag{6}$$

where $\phi = \phi(\vec{x}_1, t_1; \ldots; \vec{x}_n, t_n)$ and

$$\phi_{,\mu_i} = \frac{\partial \phi}{\partial x_i^\mu}. \tag{7}$$

It then follows by the standard variational procedures that the field equations are

$$\left(\Box_i^2 + m_i^2\right) \phi = 0 \tag{8}$$

for all i and the total stress–energy–momentum tensor in terms of ϕ and ϕ^* is

$$T_{\mu\nu} = \sum_i \frac{1}{2} \left[\phi_{,\mu_i}^* \phi_{,\nu_i} - g_{\mu\nu} \left(\phi^{\cdot \sigma_i} \phi_{,\sigma_i} - m_i^2 \phi^* \phi \right) + \phi_{,\mu_i} \phi_{,\nu_i}^* \right]. \tag{9}$$

Each individual part of T_ν^μ will satisfy a conservation relation since ϕ satisfies the field equations (8).

One thus has

$$\left(T_\nu^\mu\right)_{(i)} W_{(i)}^\nu = \lambda_{(i)} W_{(i)}^\mu \tag{10}$$

where $W_{(i)}^\mu$ gives the four-velocity of particle i. When the wave function is entangled the individual energy–momentum tensors are a function of all the particle coordinates and the $W_{(i)}^\mu$ shows non-local coupling effects even in the non-interacting particle case. The above may also be extended to include an external electromagnetic field (see [11] for details). Once the state ϕ is given, the stress–energy–momentum tensor for each particle can be calculated along with its eigenvalues and eigenvectors. As we showed in [10], for each particle one finds a pair of eigenvectors, one of which is time-like and the other space-like. The time-like vector and its eigenvalue determine the flows of energy and the density, respectively. As we have also shown elsewhere, the individual particle velocities using equation (4) are given by

$$(v^k)_i = \left(\frac{S^k \pm e^{\pm\theta} P^k}{-(\partial S/\partial t \pm e^{\pm\theta} \partial P/\partial t)} \right)_i \tag{11}$$

where

$$(\sinh \theta)_i = \left(\frac{P^\mu P_\mu - S^\mu S_\mu}{2 P^\mu S_\mu} \right)_i \tag{12}$$

It would therefore appear that in the non-interacting field case one can calculate many-particle trajectories. However, for the case of entangled wavefunctions (which imply non-local correlation of the individual particle trajectories) the prescription given thus far is not complete.

In order to calculate specific trajectories some further structure is required to specify the coordination of the particles in time. Consider the entangled two-particle wave function, defined on a four-dimensional configuration spacetime

$$\Psi\left(x_1^{(1)}, x_0^{(1)}; x_1^{(2)}, x_0^{(2)}\right) = \phi_a\left(x_1^{(1)}, x_0^{(1)}\right) \phi_b\left(x_1^{(2)}, x_0^{(2)}\right) + \phi_b\left(x_1^{(1)}, x_0^{(1)}\right) \phi_a\left(x_1^{(2)}, x_0^{(2)}\right) \tag{13}$$

which may be taken to describe the motion of two non-interacting particles, each confined to one dimension ($x_1 = z$), and we shall write $x_0 = t$. In order to calculate the world lines of a particular configuration of particles four coordinates defining a point in the configuration spacetime must be specified. In the relativistic case, with two time coordinates, $\left(t^{(1)}, t^{(2)}\right)$, the combination of this initial point and the wave function (13) is not yet sufficient to define a unique trajectory. One also needs to specify an extra relationship between $\left(x^{(1)}, t^{(1)}\right)$ and $\left(x^{(2)}, t^{(2)}\right)$. That is, using our prescription, although $\partial x_1^{(1)}/\partial t^{(1)}$ and $\partial x_1^{(2)}/\partial t^{(2)}$ can be calculated, to update the particle positions one then needs to calculate $\left(\partial x_1^{(1)}/\partial t^{(1)}\right) \triangle t^{(1)}$ and $\left(\partial x_1^{(2)}/\partial t^{(2)}\right) \triangle t^{(2)}$ and the problem is how to choose the relative sizes of $\triangle t^{(1)}$ and $\triangle t^{(2)}$

given that different choices will, with wave functions like (13), yield different trajectories. One needs a covariant prescription.

3. Lorentz-invariance

The light cone is a universal invariant local structure which may be employed to give the desired covariant theory. The invariance of the light cone can be characterized by the eigenvalues and eigenvectors of the Lorentz transformation. For simplicity we discuss the case of one spatial dimension. Writing

$$\begin{bmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix} \tag{14}$$

we find

$$a\cosh\alpha - b\sinh\alpha = \lambda a \tag{15}$$

$$-a\sinh\alpha + b\cosh\alpha = \lambda b \tag{16}$$

hence

$$\frac{a}{b} = \frac{\sinh\alpha}{\cosh\alpha - \lambda} = \frac{\cosh\alpha - \lambda}{\sinh\alpha} \tag{17}$$

and

$$\lambda = \cosh\alpha \mp \sinh\alpha = e^{\mp\alpha} \tag{18}$$

for $a/b = 1$ we have $\lambda = e^{-\alpha}$ and for $a/b = -1$ we have $\lambda = e^{+\alpha}$. Eigenvectors are null vectors which are scaled by $e^{\mp\alpha}$ after Lorentz transformation. For relative velocity v, α is defined by

$$v = \tanh\alpha \tag{19}$$

and we have [12]

$$\frac{1+v}{1-v} = \frac{\cosh\alpha + \sinh\alpha}{\cosh\alpha - \sinh\alpha} = e^{2\alpha} \tag{20}$$

and

$$e^{+\alpha} = \sqrt{\frac{1+v}{1-v}} \tag{21}$$

$$e^{-\alpha} = \sqrt{\frac{1-v}{1+v}}. \tag{22}$$

This can easily be generalized to any homogeneous Lorentz transformation.

We now proceed as follows. At the location of each particle construct the local null surfaces (in Minkowski spacetime this will be extended over spacetime with coordinates defined as $l = t + z$, $n = t - z$, $m = x - \mathrm{i}y$, $\bar{m} = x + \mathrm{i}y$). A step of ε in the rest frame of the particle becomes

$$\triangle v = \varepsilon\sqrt{\frac{1+v}{1-v}} \tag{23}$$

$$\triangle u = \varepsilon\sqrt{\frac{1-v}{1+v}}. \tag{24}$$

Since $\triangle u \triangle v = \triangle t^2 - \triangle z^2 = \varepsilon^2$, ε is the proper time step along the particle trajectories. At the position of particle 1(2) moving with velocity $v_{1(2)}$ with respect to a given inertial frame:

- Advance the position of particle 1 by

$$\triangle v_1 = \varepsilon \sqrt{\frac{1 + v_1}{1 - v_1}} \tag{25}$$

$$\triangle u_1 = \varepsilon \sqrt{\frac{1 - v_1}{1 + v_1}} \tag{26}$$

and advance the position of particle 2 by

$$\triangle v_2 = \varepsilon \sqrt{\frac{1 + v_2}{1 - v_2}} \tag{27}$$

$$\triangle u_2 = \varepsilon \sqrt{\frac{1 - v_2}{1 + v_2}}. \tag{28}$$

- Recalculate the velocities at $u_1 + \triangle u_1$, etc, and repeat the process.

If one changes the inertial frame with relative velocity $u = \tanh \beta$ then

$$e^{\mp \alpha_{1,2}} \rightarrow e^{\mp \alpha_{1,2} + \beta} \tag{29}$$

which corresponds to the relativistic formula for the addition of velocities

$$\frac{v_{1,2} + u}{1 + v_{1,2}u} = \frac{\tanh \alpha_{1,2} + \tanh \beta}{1 + \tanh \alpha_{1,2} \tanh \beta} = \tanh(\alpha_{1,2} + \beta). \tag{30}$$

In whatever inertial frame one chooses, one considers each one of the particles at rest with the appropriate $\triangle u$, $\triangle v$. The formulation is the same for all inertial frames and hence is covariant. One has taken equal proper time steps for each particle. In employing this method we note that for calculational purposes it is possible to stick with the frame coordinates since

$$\frac{\triangle u + \triangle v}{2} = \triangle t = \varepsilon \cosh \alpha \tag{31}$$

$$\frac{-\triangle u + \triangle v}{2} = \triangle x = \varepsilon \sinh \alpha. \tag{32}$$

Quantum mechanics provides a given state specified on the relativistic configuration spacetime (of $4n$ dimensions) for the many-particle system. In order to provide flow-lines in the relativistic configuration spacetime we need both the eigenvectors of the stress–energy–momentum tensor and a rule for coordinating the spacetime points on the trajectories of the particles. In preparing a given state both the space and the time coordinates of the particles are hidden variables in the sense that, unlike in classical mechanics, we cannot control their values.

4. Illustrative simple model: bound and separated, entangled particles

In order to illustrate our approach we shall consider a system consisting of two massive spin-zero bosons of equal mass, moving in one dimension, with coordinates $\left(z^{(1)}, t^{(1)}\right)$ and $\left(z^{(2)}, t^{(2)}\right)$ and described by the two-particle, multi-time Klein–Gordon equation

$$(\square_1 + \square_2) \, \Psi \left(z^{(1)}, t^{(1)}, z^{(2)}, t^{(2)}\right) = m^2 \Psi \left(z^{(1)}, t^{(1)}, z^{(2)}, t^{(2)}\right) \tag{33}$$

where we have taken $\hbar = c = 1$.

In our simple model each particle is bound in a separate one-dimensional potential well, the one dimension is taken to be the z axis, but each well is located at widely separated positions on the y axis. Apart from enabling the particles to be described as widely separated in space the y coordinate plays no further role and will not be explicitly referenced in the wave functions. The two confining potentials are taken to be stationary in the Σ frame in which we specify the wave function as the entangled state

$$\Psi\left(z^{(1)}, t^{(1)}, z^{(2)}, t^{(2)}\right) = \phi_g\left(z^{(1)}, t^{(1)}\right)\phi_e\left(z^{(2)}, t^{(2)}\right) + \phi_e\left(z^{(1)}, t^{(1)}\right)\phi_g\left(z^{(2)}, t^{(2)}\right) \tag{34}$$

where

$$\phi_g(z, t) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi z}{L}\right)e^{i\omega_g t} \tag{35}$$

is the ground state of the well and

$$\phi_e(z, t) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi z}{L}\right)e^{i\omega_e t} \tag{36}$$

is the first excited state. The frequencies are given by

$$\omega_g = \sqrt{\left(\frac{\pi}{L}\right)^2 + m^2} \tag{37}$$

$$\omega_e = \sqrt{\left(\frac{2\pi}{L}\right)^2 + m^2}. \tag{38}$$

In our approach the individual particle velocities can be calculated according to equation (11). The velocity of each particle depends on all of the coordinates $\left(z^{(1)}, t^{(1)}, z^{(2)}, t^{(2)}\right)$, but as discussed above, the specification of a particular set of values of these coordinates along with a velocity formula, although necessary, is not sufficient to produce a unique trajectory in the four-dimensional spacetime. Our covariant formulation provides the necessary extra condition to produce a unique Lorentz-invariant particle trajectory for the system in the four-dimensional spacetime. This trajectory can be projected into a pair of particle trajectories in any inertial frame. Changing inertial frame simply amounts to a passive recoordinatization of the configuration spacetime and the invariant trajectory. The points $(z^{(1)_j}, t^{(1)_j})$ on the trajectory of particle 1 are correlated with the points $(z^{(2)_j}, t^{(2)_j})$ on the trajectory of particle 2 in a unique and Lorentz-invariant manner. That is to say that if the calculation is carried out using a different inertial frame, Σ', then the transformed wave function and transformed initial coordinates produce trajectories which are the Lorentz transforms of those in Σ, and the individual points on the particle trajectories are correlated in the same way. There is now a unique velocity at each point in the four-dimensional configuration spacetime of the two-particle system, hence there is only one trajectory passing through each point.

Let us consider a specific case, if $t^{(1)_0} = t^{(2)_0} = 0$, the particles start from equal times (in the Σ frame), the three-velocities of the particles in Σ are both zero and the particles remain coordinated at equal times in this frame. This is shown in figure 1, where the initial point is taken as (1.0, 0.0, 2.0, 0.0) (in the Σ frame) and the coordination of the points on the trajectories of each particle is indicated by the numbering. In figure 2 the initial point was taken as (1.0, 1.0, 2.0, 0.0) (in the Σ frame), the three-velocities are no longer zero and the points do not remain coordinated at a constant time difference.

Covariance is assured since we use the proper times of the particles to establish the coordination of the points on the individual particle trajectories. One may choose an ensemble of trajectories that cross the $t^{(1)} = t^{(2)} = 0$ hyperplane and this would correspond with
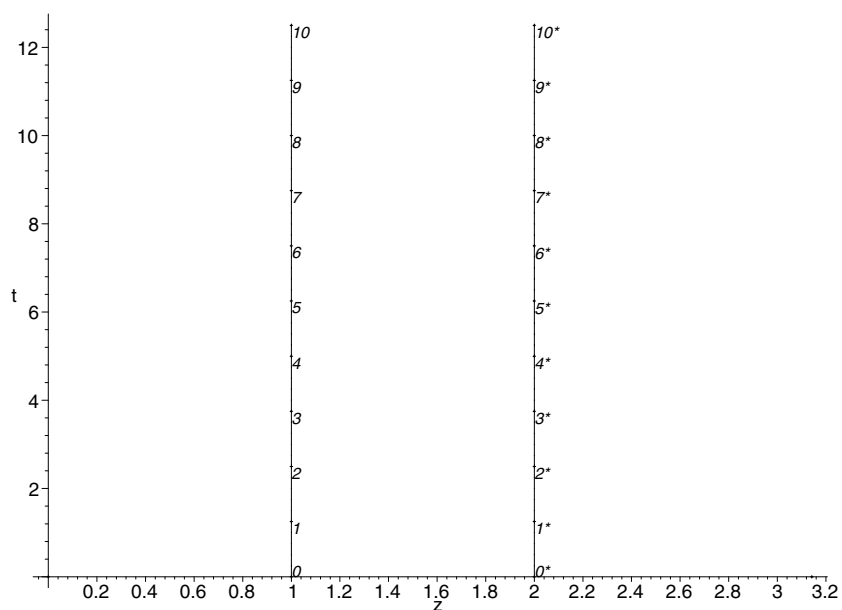
Figure 1. The initial point is taken as (1.0, 0.0, 2.0, 0.0) (in the Σ frame) and the coordination of the points on the trajectories of each particle is indicated by the numbering.
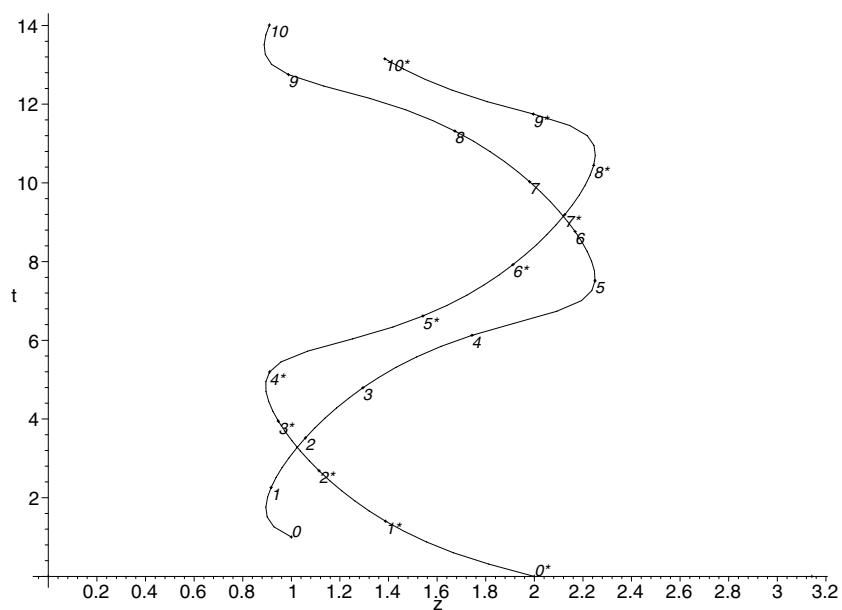

Figure 2. The initial point was taken as (1.0, 1.0, 2.0, 0.0) (in the Σ frame), the three-velocities are no longer zero and the particles do not remain coordinated at a constant time-difference.

the set of trajectories which are at equal times, namely $t = 0$, in Σ. One is not bound to make such a choice of initial points but once such an ensemble is agreed upon, all observers would calculate the same trajectories. This means that, given the hidden variables, the outcomes of

any measurements are completely determined, irrespective of the frame of reference in which the motion is described. In the context of this simple model one can imagine measuring the energy of the particles by removing the confining potentials and allowing the particles to run out. A time-of-flight measurement then reveals the energy of the particle. Consider the case in which the potentials are removed at equal times $t^{(1)} = t^{(2)} = \tau$ in Σ. If the particles cross the $t^{(1)} = t^{(2)} = 0$ hyperplane then they reach the $t^{(1)} = t^{(2)} = \tau$ hyperplane at equal proper times. The trajectories, and hence the measurement outcomes at this time, can be predicted from the no-crossing rule. We find that if $\left| z^{(2)} \right| > \left| z^{(1)} \right|$ then particle 2 moves away quickly and thus is the excited particle whereas particle 1 moves slowly and is the ground state particle. On the other hand if $\left| z^{(2)} \right| < \left| z^{(1)} \right|$ then the outcome is reversed; particle 1 moves away quickly and hence is the excited particle while particle 2 moves slowly and is the ground state particle. Given the initial point in configuration spacetime, a unique outcome is guaranteed. As we have emphasized above, adopting a different frame amounts to a passive recoordinatization of the spacetime. Different outcomes would only be predicted if the initial point in the configuration spacetime was taken to be different and there is no reason to change this point just because a different frame for the description of the experiment is adopted.

5. Conclusion

We have shown that there is no necessary contradiction between relativistic quantum mechanics and a Lorentz-invariant hidden-variable theory based on particle trajectories. The theory that we propose requires one extra Lorentz-invariant rule which produces unique flow-lines in the many-particle relativistic configuration space. The extra rule applies equally to bosons and fermions. As is usual in the de Broglie–Bohm approach to quantum mechanics, measurement plays no special role; rather the introduction of a measuring device merely enlarges the configuration space in which the whole system is described.

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